

# The origin of entropy production in spacetime thermodynamics

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## Abstract

We find that the ambiguity term of approximate Killing vector field is responsible for the entropy production term. Without the ambiguity term, pure Einstein theory and  $f(R)$  satisfy the relation of thermodynamic equilibrium. Considering such an ambiguity term of approximate Killing vector field, we can get the entropy production term and the entropy in  $f(R)$  with a form defined by Jacobson. In pure Einstein theory, the shear term is the only geometric contribution of entropy production term, while in  $f(R)$  it can also contribute. We believe our approach and conclusion can be generalized to other gravity theory.

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## I. INTRODUCTION

The discovery of black hole entropy and Hawking radiation in 1970s [1] implies a profound connection between gravitation and thermodynamics. Almost twenty years ago, Jacobson [2] proposed to interpret this connection by reversing the logic and deriving the Einstein's equations from a thermodynamic equation of state with two assumptions: proportionality of entropy and area for all local Rindler horizons, and the Clausius relation. The first assumption is nothing but the expression of the holographic principle [3][4]. In 1998, Maldacena gave a gorgeous conjecture, namely the AdS/CFT correspondence [5], which is a direct manifestation of the holographic principle. It was suggested that gravitation is induced by a quantum field theory in lower dimensions which to a large extent support the idea of gravitation on the macroscopic scale as a manifestation of the thermodynamics of the vacuum state. Furthermore, more confidence about gravity being emergent rather than fundamental was recently inspired in [6][7][8][9][10].

In [11], Jacobson treated  $f(R)$  theory as non-equilibrium thermodynamics of spacetime, therefore, an entropy production term is required to keep the entropy balance relation. The term is an outcome of the non-vanishing expansion at  $p$  which is related to a local boost dependence quantity. In 2008, Elizalde and Silva [12] proposed an alternative approach, with local thermodynamic equilibrium maintained, using the idea of “local-boost-invariance” introduced in [13]. In other words, local boost dependence quantity is erased under the boost-invariant truncation, so expansion at  $p$  vanishes, without any entropy production term when we repeat Jacobson's derivation. Brustein and Hadad [14] showed that the equations of motion of generalized theories of gravity are equivalent to the thermodynamics relation  $\delta Q = T\delta S$ . Their proof relies on extending previous arguments by using a more general definition, namely Noether charge entropy.

In this paper, we focus on the origin and feature of the entropy production. We show that it is a consequence of the  $o(x^3)$  ambiguity in an approximate boost Killing vector field. If we suspend this ambiguity first, the entropy production will vanish, and vice versa. In order to research the thermodynamics of different gravity models, we assume the validity of those gravity field equations, and then use them to derive the form of entropy. In  $f(R)$  gravity, if the “boost variant” (a Lie derivative of  $f(R)$  along Killing vector at a point  $p$  in the spacetime) is nonzero, the entropy production is a function of the “boost variant” and

non-vanishing expansion at  $p$ . In pure general relativity (GR), we can not adopt such approach. The shear term from Raychaudhuri equation can include the entropy production.  $f(R)$  theory can also accept the shear term, instead of non-vanishing expansion term, as the entropy production term. At last, we may find that the shear term being the entropy production term is more general. Conditions of pure GR and  $f(R)$  theory analyzed, we believe our method and results can also be generalized to other gravity theories.

## II. APPROXIMATE KILLING VECTOR FIELD

In this section, we review Jacobson's work briefly and set the background of our work. First, we can describe spacetime in a vicinity of a free-falling local observer  $p$  as flat through the equivalence principle. Then, we choose a local 2-surface element  $B$  including  $p$  and perpendicular to the worldline of  $p$ . The boundary of the past of  $B$  is defined as "local Rindler horizon" [horizon entropy], whose generators are a congruence of null geodesics with vanishing expansion and shear. Therefore, the local Rindler horizon reaches equilibrium at  $p$ . Considering the local Rindler horizon is actually a causal horizon, we can introduce an entropy  $S$ , measuring the degrees of freedom beyond it. According to the holographic principle,  $S$  is proportionate to the area elements of the horizon.

The definition of the heat flux and temperature is related to an approximate boost Killing vector field  $\chi^a$ .  $\chi^a$  generating boosts orthogonal to  $B$  the causal horizon. It vanishes at  $p$ , its flow invariant at the tangent plane  $B_p$ . Its covariant derivative  $\chi_{a;b}$  is a timelike antisymmetric tensor orthogonal to  $B_p$ . We normalize  $\chi^a$  by  $\chi_{a;b}\chi^{a;b} = -2$ . In a common curved spacetime, no Killing vectors exist. we can only solve the Killing equation  $\nabla^a\chi^b + \nabla^b\chi^a = 0$  with this "initial data" out to some order in the neighborhood of  $p$ . The equation

$$\nabla_a\nabla_b\chi^a = R_{ab}\chi^a, \quad (1)$$

is equal to the Killing equation. In Riemann normal coordinates  $\{e_\mu^a\}$  based at  $p$ , the zeroth and the first order parts of  $\chi^a$  are resolved by initial conditions, while the second order part vanishes according to the Killing's equation (compare both sides of Eq.1 to get this conclusion). Generally, the equation cannot be satisfied at third order, so we still have a  $o(x^3)$  ambiguity in Killing vector  $\chi^a$ , which would influence the integrability of equation 1 (refer to Weinberg's text book [15] for detail). We choose the direction of  $\chi^a$  to be future

pointing on the causal horizon. In flat spacetime,  $\chi^a$  can be defined as  $\chi^a = -\lambda k^a$ , where  $k^a$  is the horizon tangent vector and  $\lambda$  is a negative affine parameter that is increasing along the horizon and vanishes at  $p$ . Here, we set light-cone coordinates as  $k^a = (e_0^a + e_3^a)/\sqrt{2}$ ,  $l^a = (e_0^a - e_3^a)/\sqrt{2}$ ,  $x^a = e_1^a$ ,  $y^a = e_2^a$ , which satisfy

$$g_{ab}k^a l^b = -1, \quad g_{ab}k^a k^b = g_{ab}l^a l^b = 0. \quad (2)$$

With the third order ambiguity in curved spacetime,  $\chi^a$  should be

$$\chi^a = -\lambda k^a + o(x^3). \quad (3)$$

and therefore Eq.1 becomes

$$\nabla_a \nabla_b \chi^a = R_{ab} \chi^a + o(x). \quad (4)$$

After defining the local Rindler horizon and approximate Killing vectors, it is natural to define the heat. The heat is the mean flux of the boost energy current across the horizon measured by a uniformly accelerated observer hovering inside the horizon:

$$\delta Q = \int_H T_{ab} \chi^a \epsilon^b \quad (5)$$

$T_{ab}$  is the expectation value of the matter stress tensor, and  $\epsilon^b$  is the area of each cross section element of  $H$ . The integration is over a short pencil of horizon generators of  $H$ , and  $\lambda$  contained in  $\chi^a$  represents the evolution of those generators. Since it is an equilibrium state at  $p$ , the metric and, further, its conjugate  $T_{ab}$  are also approximately stable. It means the expectation value of the matter stress tensor do not change over a sufficiently small  $\lambda$  or, equivalently, quantum transitions terminate. Without loss of generality, we choose  $T_{ab}|_p = T_{ab}|_{\lambda=0}$ .

According to the Unruh effect and Rindler coordinates, we know acceleration tends to diverge as observer approaches the horizon. Since both of the temperature and heat flux are proportional to the acceleration, they diverge the same rate. Thus, we can choose the acceleration to be unit and the temperature measured by the observer to be  $T = \hbar/2\pi$ .

### III. THE CALCULATION OF $F(R)$

In Jacobson's paper, he reversed the logic, using two hypotheses, universal entropy density of the horizon and the Clausius relation in vicinity of  $B$ , to derive the Einstein equation.

Some gravity theories, like  $f(R)$  gravity, may need an entropy production rate to keep the entropy balance relation. However, we wonder that the entropy may contain un-physical quantities which will cause the entropy production. In [12][14], Wald entropy was introduced immediately, so the un-physical quantities were truncated.

In this section, we postulate the  $f(R)$  gravity field equation first, and then ascertain the form of entropy affecting the dynamics. Considering the Killing equation and approximate Killing vector field, we will find the entropy production rate arising from the ambiguity term.

The equation of motion of  $f(R)$  is

$$F(R)R_{ab}(g) - \frac{1}{2}f(R)g_{ab} - \nabla_a \nabla_b F(R) + g_{ab} \square F(R) = 8\pi G T_{ab}. \quad (6)$$

Put it into the heat, it follows that

$$\delta Q = \frac{1}{8\pi G} \int (F R_{ab} - \nabla_a \nabla_b F) \chi^a k^b d\lambda dA. \quad (7)$$

For simplicity, we leave the ambiguity term  $o(x^3)$  of Eq.1 alone and pick it up afterward. Terms with  $g_{ab}$  vanish since  $g_{ab}$  contracted with  $\chi^a k^b$  leads to  $-\lambda k_a k^a = 0$  in light-cone coordinates. We then evaluate the equation at the leading terms in  $\lambda$ ,

$$\frac{\delta Q}{T} = \frac{1}{4\hbar G} \int (F|_p R_{ab} - \nabla_a \nabla_b F|_p) \chi^a k^b d\lambda dA = \frac{1}{4\hbar G} \int (F|_p \nabla_a \nabla_b \chi^a - \nabla_a \nabla_b F|_p \chi^a) k^b d\lambda dA. \quad (8)$$

The Killing equation shows  $\nabla_b \chi_a$  is antisymmetric, so we can use Stoke's theorem and obtain

$$\frac{\delta Q}{T} = \frac{1}{4\hbar G} \left\{ \int l_a F|_p k^b (\nabla_b \chi^a) dA|_0^{d\lambda} + \int \lambda \ddot{F}|_p d\lambda dA \right\} = \frac{1}{4\hbar G} \int (\theta F|_p + \lambda \ddot{F}|_p) d\lambda dA. \quad (9)$$

Here comes a contradiction that the integrand contain  $\lambda$  at beginning, while the first term at the last step do not. In other words,  $\theta F|_p$  has a term of less order  $\theta F|_p = \theta|_p F|_p + \lambda \dot{\theta}|_p F|_p$ .

It seems that  $\theta|_p$  should be zero by comparing the two sides's order of Eq.9, therefore

$$\frac{\delta Q}{T} = \frac{1}{4\hbar G} \int (\lambda \dot{\theta}|_p F|_p + \lambda \ddot{F}|_p)$$

Let's check the value of  $\theta|_p$  in another way. In Jacobson's paper, the entropy change is  $\delta S = \alpha \int (\theta F + \dot{F}) d\lambda dA$ , where  $\alpha = \frac{1}{4\hbar G}$ . It can be extracted as

$$\delta S = \alpha \int [(\lambda \dot{\theta}|_p F|_p + \lambda \ddot{F}|_p) + (\lambda \theta|_p \dot{F}|_p + \theta|_p F|_p + \dot{F}|_p)] d\lambda dA. \quad (10)$$

and the entropy balance relation is  $dS = \delta Q/T + d_i S$ ,  $\dot{F}|_p$  is the "boost variant" which mentioned in the introduction. The Raychaudhuri equation is

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} - R_{ab}k^a k^b. \quad (11)$$

the shear term is required to be zero at  $p$ . Comparing these equations, we will obtain the entropy production term

$$d_i S = \alpha \int (\lambda \theta|_p \dot{F}|_p - \frac{\lambda}{2} \theta^2|_p + \theta|_p F|_p + \dot{F}|_p) d\lambda dA.$$

However, the entropy production term is supposed to vanish at  $p$ , the equilibrium point, so the rate should be of order  $\lambda$ . It requires  $\theta|_p F|_p + \dot{F}|_p = 0$  and  $\theta|_p = 0$

$$\frac{\delta Q}{T} = \alpha \int (\lambda \dot{\theta}|_p F|_p + \lambda \ddot{F}|_p) d\lambda dA. \quad (12)$$

This equation is directly related to the equation of motion of  $f(R)$ .

Now, if we use  $\chi^a = -\lambda k^a + o(x^3)$  in Eq.7, then the integrand will have  $o(x) \sim \nabla \nabla o(x^3)$  which has the same order with other terms.  $o(x) \sim \nabla \nabla o(x^3)$  consist of two parts: one is from Eq.4, the other is from using the Stokes theorem. The ambiguity term  $o(x)$  act as  $\lambda \theta|_p \dot{F}|_p$ , which also vanishes at  $p$  and is irrelevant to the equation of motion. Substituting  $\dot{F}|_p = -\theta|_p F|_p$  to this term, we will get  $-\alpha \lambda \theta|_p^2 F|_p$  parallel with the entropy production density derived by Jacobson.

One thing need attention is we only use  $\theta|_p F|_p + \dot{F}|_p = 0$  in this part, not each of these two terms to be zero. It means that they are not related to the the equation of motion.  $\dot{F}|_p = F'(R) k^a R_{,a}$  is a “boost variant”, which is dependant on the choice of coordinates and so is  $\theta|_p$  which is restricted by  $\dot{F}|_p$ . Hence, we find the entropy production is related to boost.

#### IV. COMPARE WITH GR

In this section, we discuss the situation of pure GR under a similar procedure. Replace the equation of motion of  $f(R)$  with Einstein equation  $R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}$ , we immediately get

$$\frac{\delta Q}{T} = \frac{1}{4\hbar G} \int R_{ab} \chi^a k^b d\lambda dA = \frac{1}{4\hbar G} \int (\theta + o(x)) d\lambda dA \quad (13)$$

here  $\theta|_p = 0$  is demanded for the reason that both sides of the equation should have the same order. Without  $o(x)$ , we can get the state equation of equilibrium positively. However, if we consider ambiguity term, we can not formulate the entropy production term since expansion at  $p$  vanishes. However, in Raychaudhuri equation (Eq.11) there is a shear term  $\sigma^2 = \sigma^{ab} \sigma_{ab}$ . If the shear term is not required to be zero, it can be used to describe

the entropy production term such as  $\alpha\sigma^{ab}\sigma_{ab}$ . This is coincident with Jacobson's remarks. Actually, entropy production term comprising  $\theta|_p$  is not indispensable, although  $\theta$  can be limited by  $\dot{f}|_p$ . If, in  $f(R)$  gravity, we employ the condition of nonzero shear like in pure GR, we will also keep  $\theta|_p = 0$ . It may, from another aspect, give the reason why the thermodynamic relation  $\delta Q = T\delta S$  hold in generalized theories of gravity.

## V. CONCLUSION

These two situations analyzed above indicate that the entropy production term comes from the ambiguity term of the approximate Killing vector. We do need it for the entropy balance relation rather than the field equation. From this point of view, we find the entropy production term is not related to the equation of motion. This ambiguity term can also be constructed of nonvanishing  $\sigma^2$  or other quantities. It can be seen that our method is universal, therefore we make the conclusion that entropy production term is caused by the ambiguity term of the approximate Killing vector field. Furthermore, we believe that the entropy production term can also be described by nonvanishing  $\sigma^2$ , since Raychaudhuri equation is used generally on null geodesics distortion caused by gravity.

## VI. ACKNOWLEDGMENTS

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